Dynamical exponent of rough surface with quenched noise

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Dynamical behavior of the directed percolation model is investigated in detail. We find that the width of the interface has two scaling regions. By introducing two kinds of growth events and a proper definition of time, the first scaling region is shown to describe the transient process and the second one the asymptotic one. Moreover, we find an interesting phenomenon that at $q=0$ the exponential increasing of the width of the interface $W(L,t)$ is subject to a ''periodic'' oscillation of time t. This periodic behavior is weakened by increasing q, and it eventually disappears for sufficiently large q. [S1063-651X(97)15602-1]

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For decades, due to its importance in many fields, such as the motion of liquids in porous media, fronts of fire, and growth of bacterial colonies, the investigation of rough surfaces and interfaces has attracted much attention, and considerable progress has been made in understanding the properties in these processes [1–5]. Despite the diversity of physical phenomena, rough surfaces and interfaces can frequently be described quite well in terms of the concepts of fractal geometry. Most rough surfaces and interfaces appear to exhibit self-affine scaling over a significant range of length scales.

In general, a d-dimensional self-affine fractal, described by a single-valued function $h(x,t)$, evolves in a $(d+1)$-dimensional medium. The self-affine interface can be characterized by the rms surface width $W(L,t)$, defined as

$$W(L,t)=\{\langle[h(\vec{x},t)-\langle h(\vec{x},t)\rangle]^2\rangle^{1/2}\}, \tag{1}$$

where $h(\vec{x},t)$ is the height of the interface at time t, and the angular brackets denote average over space and different realizations of disorder. It is believed that $W(L,t)$ obeys the finite size dynamical scaling law [6]

$$W(L,t)=L^{\alpha}f\left(\frac{t}{L^z}\right), \tag{2}$$

where $f(x)$ is a scaling function, $f(x)\propto$const for $x\gg 1$ and $f(x)\propto x^{\beta}$ for $x\ll 1$, and z is the dynamical factor with $z=\alpha/\beta$. Combining them, we find

$$W(L,t)\sim\begin{cases} L^{\alpha} & (t\gg L^z) \\ t^{\beta} & (t\ll^z) \end{cases}, \tag{3}$$

where α is the roughness exponent and β the dynamical exponent, which characterize the static and dynamic properties of the rough surface, respectively.

Two main classes of disorder, which affect the motion of the interface and lead to its roughening, have been discussed in the literature. The first, called ''annealed'' noise, depends only on time. The second, ''quenched'' disorder, is frozen in the medium. Continuous equations, such as the Kardar-Parisi-Zhang (KPZ) equation [7], have been remarkably successful in describing roughening in the case of annealed disorder. Karder *et al.* used a dynamical renormalization group method to analyze the KPZ equation, and found in $1+1$ dimensions, $\alpha=\frac{1}{2}$ and $\beta=\frac{1}{3}$. Some models, such as the Eden model, the ballistic model, and the restricted solid-on-solid model, etc., give the results near the values of the KPZ equation [6,8,9]. But some experiments, such as the motion of liquids in a porous medium, and the growth of bacterial colonies [3,5], where the disorder is quenched, give the exponents anomalously larger than the ones of the KPZ equation. They give α ranging from 0.7 to 0.9. To understand this phenomenon, the directed percolation (DP) model was proposed [10,11]. In the DP model, the roughness exponent $\alpha\simeq 0.63$ is anomalously large and close to the value of the experiments. Up to now, the DP model has been considered as a universal class different from the KPZ class.

The DP model used in this paper was proposed by Tang and Leschhorn [11]. On a square lattice of edge L (with periodic boundary condition), we assign a random pinning force $f(\vec{r})$ uniformly distributed in the interval [0,1] to every cell. For a given applied pressure q $(1\geqslant p\geqslant 0)$, we can divide the cells into two groups, free cells with $f(\vec{r})\leqslant p$ and blocked cells with $f(\vec{r})>p$. Obviously, the density of the blocked cells on the lattice is $q=1-p$. Under the solid-on-solid (SOS) condition, the interface is determined by a set of integer column heights h_i, $i=1,\ldots,L$. At $t=0$, we assume that the initial interface is flat, that is, $h_i=0$, $i=1,\ldots,L$. The growth event is defined as follows. We randomly select a column, say i, and compare its height with those of the two adjoint columns $i-1$ and $i+1$, if $h_i\geqslant\min\{h_{i-1},h_{i+1}\}+2$; the site that is the smaller of the two adjoint columns is incremented by one unit (in the case of a tie, we choose one of the two with equal probability). In the opposite case, $h_1,<\min\{h_{i-1},h_{i+1}\}+2$, column i advances by one unit if the cell to be occupied is a free cell. Otherwise no growth event takes place. In this model, the time unit is defined as one growth attempt. In numerical simulation, we measure time in units of L growth attempts. It is known that there exists a critical density q_c in the limit $L\to\infty$, above which

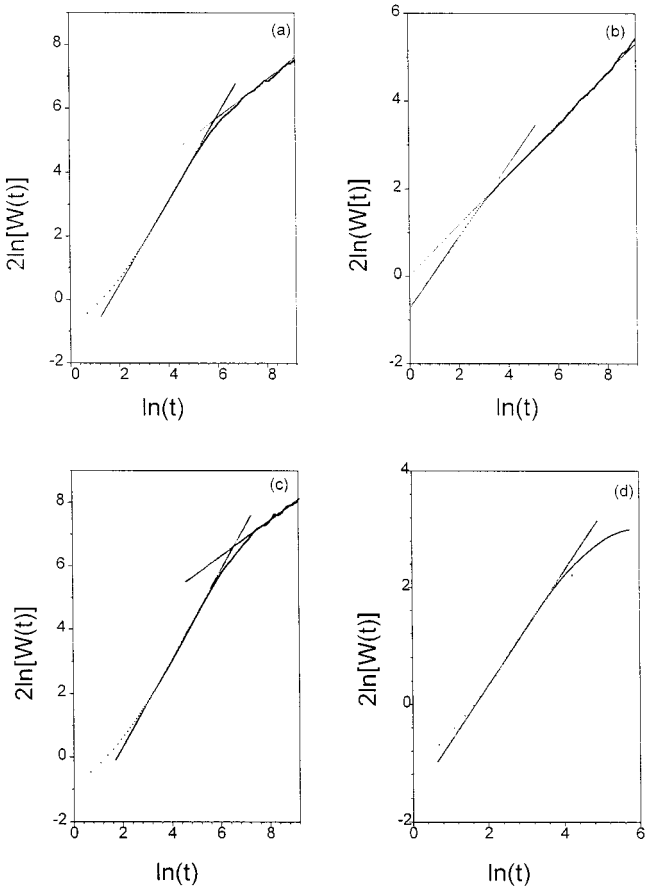

FIG. 1. Scaling of the width of rough surface vs time with $L=8192$. The curve crosses over from a power law with an exponent β_1 to another power law with an exponent β_2: (a) $q=0.49$, $\beta_1=0.649$, $\beta_2=0.304$; (b) $q=0.3$, $\beta_1=0.417$, $\beta_2=0.288$; (c) $q=0.51$, $\beta_1=0.657$, $\beta_2=0.288$; (d) $q=0.6$, $\beta_1=0.486$.

growth eventually stops; i.e., the whole interface is pinned. This critical value q_c is the threshold for the directed percolation, which is about 0.539 in this model.

In the neighborhood of q_c, the dynamic exponent β obtained by Tang and Leschhorn [10] is 0.63 both above and below q_c, while β obtained by Makse and Amaral is 0.68 in the pinning phase ($q>q_c$) and is 0.75 in the moving phase ($q<q_c$) [12]. However, no systematical investigation for the dependence of β on q has been carried out. In this paper, we will discuss this dependence in detail. We find that changing q can dramatically change β; the result is considerably different from the general picture known so far.

First, we simulate the model numerically, and the results are averaged for many realizations. In Fig. 1(a), we plot the relation between $\ln[W(L,t)]$ and $\ln(t)$ at $q=0.49$. It is found that the width $W(L,t)$ has two power law regions before it saturates to a constant. Now, we denote the exponents in the first and the second regions by β_1 and β_2, respectively. At this q, we have $\beta_1\simeq 0.649$, and $\beta_2\simeq 0.304$. In the other parts of Figs. 1, we plot the results for different q, i.e., $q=0.3$, 0.51, 0.6. From these figures, we find the scaling region characterized by β_1 is enlarged by increasing q, while the region characterized by β_2 shrinks as q increases. The second scaling region disappears when $q>q_c$ due to the fact that the interface is pinned.

Due to the self-affine fractal nature of the interface, the DP model has two kinds of correlation lengths. One is parallel to the flat reference surface, $\xi_\parallel$ and the other is vertical to the reference surface, $\xi_\perp$. $\xi_\perp$ is proportional to the width of the interface $W(L,t)$. As $q<q_c$, some sections of the interface are pinned and the growth occurs only in columns that contain free cells on the interface, and when $q>q_c$, the blocked cells can form a connected cluster spanning the whole system and the interface is pinned. Usually the average horizontal size of the pinned section is considered as $\xi_\parallel$. However, there may be another situation where $\xi_\parallel$ is the average horizontal length of sections of the interface formed by connected free cells; this situation happens for sufficiently small q. In fact, the definition of the correlation function is

$$G(\vec{r},\vec{r}',t)=\langle[h(\vec{r},t)-\langle h\rangle][h(\vec{r},t)-\langle h\rangle]\rangle. \tag{4}$$

As a conclusion, correlation length should be mainly determined by the finite cluster. Here, the finite cluster means that it cannot span the whole system. This point can be manifested by the DP model. We know that the blocked cells form a connected cluster that spans the whole system for $q>q_c$ while free cells form a connected cluster spanning the whole system only for $q<1-q_c$. Consequently, $\xi_\parallel$ is mainly determined by the pinned section for $q<1-q_c$ while by the moving section formed by free cells for $q>q_c$. It is worth mentioning that the infinite cluster does not exist either among blocked cells or among free cells for $1-q_c<q<q_c$.

According to the results of Fig. 1, the scaling form Eqs. (3) must be modified to

$$W(L,t)\sim\begin{cases} t^{\beta_1} & (t\ll\xi_\parallel^{1/z}) \\ t^{\beta_2} & (\xi_\parallel^{1/z}\ll t\ll L^{1/z}) \\ L^{\alpha} & (t\gg L^{1/z}) \end{cases}. \tag{5}$$

From Eq. (5), we know that β_1 is the exponent that characterizes the process in which growth evolves from a site into correlation length, $\xi_\parallel$, and $\xi_\parallel$ decreases when q decreases from q_c. As a result, the scaling range characterized by exponent β_1 shrinks when q decreases from q_c. The exponent β_2 characterizes the process in which the interface develops in the size from $\xi_\parallel$ to the whole system.

Furthermore, we numerically simulate the relations between $\ln[W(L,t)]$ and $\ln(t)$ for various q. From the data, we measure β_1 and β_2 for each q, and plot β_1 and β_2 versus q in Fig. 2. The range of q examined is from 0.2 to 0.72, which runs from a parameter far below the pinning-depinning point, to that far above the point. It is rather unexpected that β_1 depends on q so strongly. β_1 increases when q increases from 0.2 to 0.46. A small β_1 flat plateau is found in the range from 0.46 to 0.54. After the pinning-depinning point q_c, β_1 decreases when q increases. Therefore, the β_1-q relation shows an interesting curve. This picture tells us that β_1 is a function of density of blocked cells. A heuristic explanation can be made for this curve. The change of β_1 manifests the result of the competition between the free cells and the blocked cells. The value of β_1 is mainly determined by the connected section formed by the blocked cells or free cells that cannot span the whole system. With the increase of the probability of large connected sections, the interface becomes rougher; as a result, the exponent β_1 increases. In the

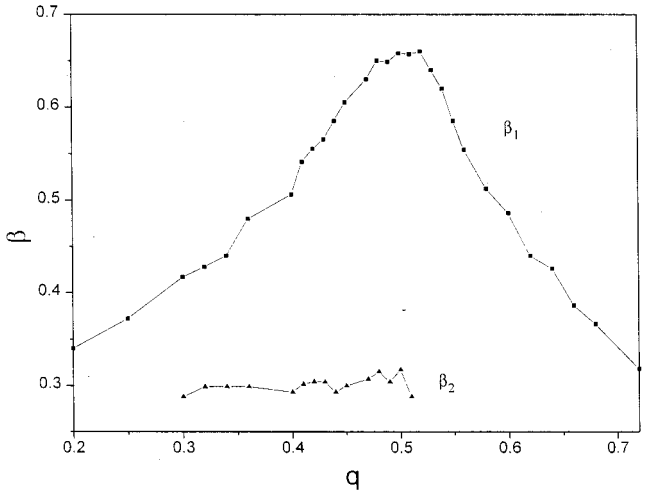

FIG. 2. Dynamical exponents β_1 and β_2 as functions of q.

first region [$q \in (0.2, 0.46)$], free cells can form the connected cluster that spans the whole system, and it is the blocked cells that influence the properties of the system. With the increase of q, larger connected sections formed by blocked cells may appear frequently, then β_1 increases with the increasing of the density of the blocked cells. In the third region [$q \in (0.54, 0.72)$], blocked cells can form the connected cluster that spans the whole system, it is free cells that are relevant to the scaling behavior, and when q increases, β_1 decreases with decreasing of the density of the free cells. Finally, in the second region, neither free cells nor blocked cells can form the cluster spanning the whole system, the influences of these two kinds of cells on the system dynamics are comparable, and the competition between these two kinds of cells makes the flat plateau of β_1 appear. Similarly, we measure the value of β_2 (it exists only in the moving phase) and plot the results also in Fig. 2. It is surprising that the behavior of β_2 is considerably different from β_1. β_2 fluctuates around 0.3 as q varies, namely, β_2 is practically a constant ($\beta_2 = 0.3$) independent of q in the permitted range of error. We consider the fact that the interface consists of connected sections of width $\gg 1$, building a driving interface on large scale. As these sections contribute to total width, we can expect that the numerical estimate of β_2 is underestimated and it is possible that the actual β_2 is a bit larger and it may be $\frac{1}{3}$ on large scales, which is the result of the KPZ equation [7].

Our numerical results are qualitatively consistent with the β value experimentally determined by Horvath and Stanley [13]. In the investigation of the growing interface during imbibition of viscous liquids in filter paper, Horvath and Stanley found (1) $\beta = 0.56 \pm 0.03$ with β independent of driving and (2) no crossover from β_1 to β_2 in the scaling regime. In comparison with our results, their experiment condition corresponds to the neighborhood of the critical value q_c, where our numerical value of β_1 is constant and the scaling region of β_2 disappears. We may expect the appearance of a crossover of the scaling region and the driving dependence of β for large driving (i.e., for small q).

In order to investigate the mechanism underlying dynamical behavior of the DP model further, let us discuss two

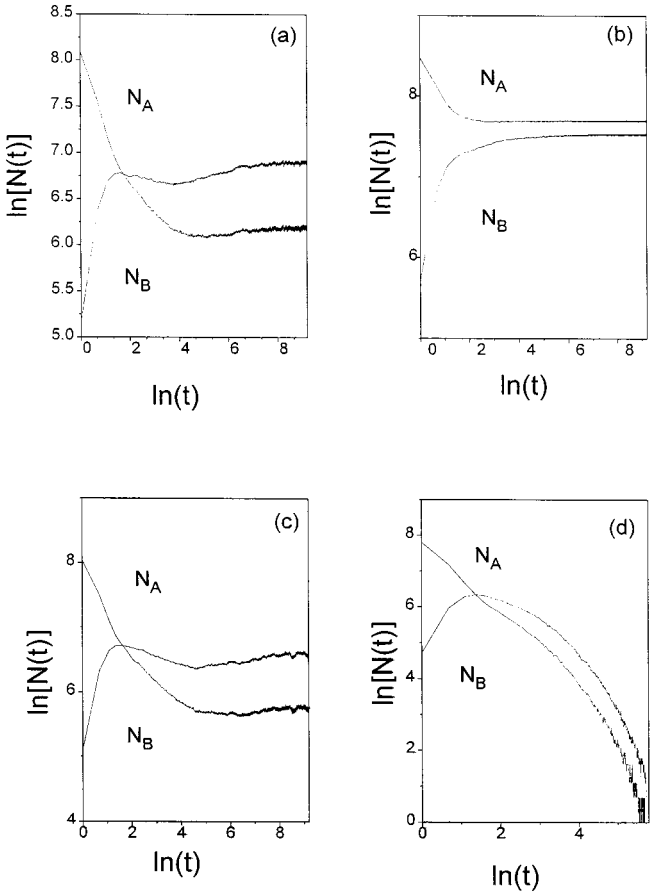

FIG. 3. N_A and N_B vs time with $L = 8192$. N_A and N_B are the numbers of the growth occurring at the site chosen or its adjoint site in certain time interval, respectively. (a) $q = 0.49$; (b) $q = 0.3$; (c) $q = 0.51$; (d) $q = 0.6$.

kinds of growth events. According to the definition of the model, the site increasing a unit, $h_{i'} = h_{i'} + 1$, may be a site we have chosen or may be a site adjoint to the site chosen; then we can define the event in which the growth occurs at the site we have chosen randomly as a type A event and the event in which the growth occurs at an adjoint site as a type B event. In the numerical simulation, we count the numbers of types A and B growth events in a time interval of L growth attempts. $N_A(t)$ denotes the number of events A in L growth attempts and $N_B(t)$ denotes the number of B. In Fig. 3, we plot $N_A(t)$ and $N_B(t)$ versus t for different q (the values of q are the same as the corresponding values in Fig. 1). From Fig. 3, we find $N_A(t)$ is dominant for small t. When t increases, $N_A(t)$ decreases while $N_B(t)$ increases for small t. In the large time region both $N_A(t)$ and $N_B(t)$ tend to certain constants, which are finite for the moving phase and zero for the pinning phase. The entire temporal behavior of the system can be understood heuristically. For the model we study, the initial surface is flat. Therefore at the beginning, the differences of height between most adjoint sites satisfy the requirement $\Delta h \leqslant 2$; $N_A(t)$ is much larger than $N_B(t)$ for small t. When growth continues, more and more height differences between adjoint sites exceed the requirement, the probability that growths occur at the adjoint site becomes larger, as a result, $N_B(t)$ increases and $N_A(t)$ decreases. However, in the case of $q < q_c$ these increasing and decreas-

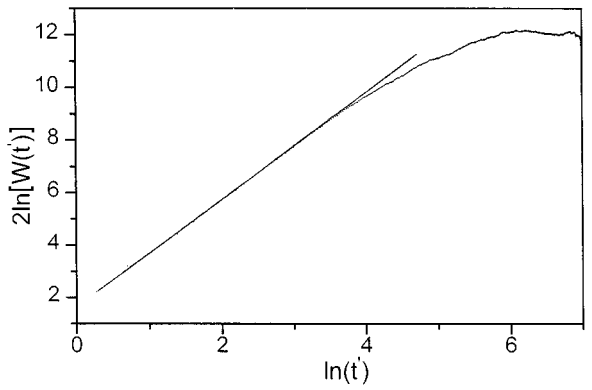

FIG. 4. Scaling of the width of rough surface vs the modified time with $q=0.539$, $L=8192$. The data are well fitted by $W\sim t^{\beta'}$ with $\beta'=1.1$.

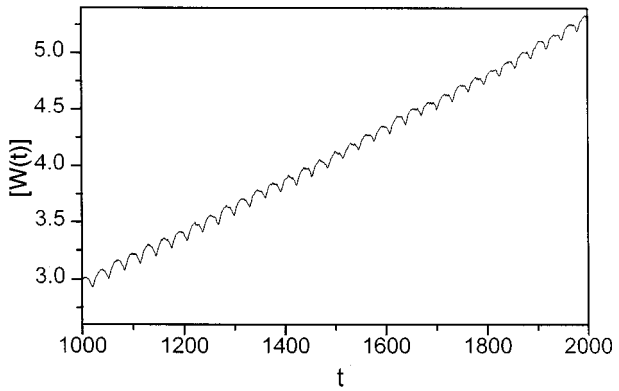

FIG. 5. Width vs time with $q=0$, $L=8192$. The periodic behavior of width added to the exponentially increasing tendency is observed; the period is $30L$.

ing tendencies saturate to an equilibrium balance in the large time region, the system reaches a steady state, and both $N_A(t)$ and $N_B(t)$ will be constants with small fluctuation.

By comparing Figs. 1 and 3, we find that the variations of $N_A(t)$ and $N_B(t)$ have close correspondence with the scaling exponents β_1 and β_2. In the region characterized by β_1, $N_A(t)$ and $N_B(t)$ change rapidly. While in the region characterized by β_2, $N_A(t)$ and $N_B(t)$ are practically constants if we average them for large number of samples. This phenomenon is independent of q, and it is the reason that β_2 is a constant, namely, β_2 is determined by the dynamic balance induced by the competition between events A and B. By investigating events A and B, we can find that β_1 is an exponent characterizing a transient process and β_2 is an exponent characterizing the asymptotic (asymptotic in the sense $\xi_{\parallel}^{1/z}\ll t\ll L^{1/z}$) behavior of the system.

Comparing the DP model [10,11] with the SOC model proposed by Sneppen [14], we observe that there are two kinds of differences between the two models. One well known difference is that the external force applied to the system in the DP model is constant, while the force is a function of time in the SOC model (in the latter case the system selects a smallest force for growth). The other is that there is one growth at every time unit in the SOC model, while there is only a growth attempt for every time unit, growth may occur or not occur at a given time unit. Now, we will focus on the latter difference, and try to link the DP model with the SOC model by modifying the time definition of the DP model. For this purpose, we define the time unit of the DP model as a true growth not a growth attempt. Therefore, under this definition, there may be many growth attempts between two successive time units, i.e., between two successive actual growth events. The SOC system will reach its critical state after a transient process, so in order to compare with the SOC model, we should investigate the DP model in the neighborhood of the transition point. Then we focus our attention on $q=0.539$. In Fig. 4, we plot $\ln[W(L,t')]$ versus $\ln(t')$, where t' is the modified time (i.e., t' is counted by true growth events, not by growth attempts). From Fig. 4, we find the dynamical exponent $\beta'[W(L,t')\sim t'^{\beta'}]$ is about 1.1 for small t' and this value is very close to the exponent $\beta_{\rm tr}$ in the transient process of the SOC model [14–16]. The transient exponent $\beta_{\rm tr}$ of the SOC system describes the dynamical behavior of the system before it reaches the critical state. In our case we also find that β_1 describes the dynamical behavior of the system during the process in which a ''perturbation'' is propagated to the size of correlation length. Under the modified definition of time the closeness of the exponent β' of the DP model with that of the SOC model $\beta_{\rm tr}$ manifests the intrinsic link between these two important models, which is worth investigating further. This fact shows that β_1 is related to $\beta_{\rm tr}$ and it is a transient dynamical exponent.

Another interesting point is that β_1 ($\beta_1\sim0.66$ at $q=0.539$) under the original definition of time is close to $\beta_{\rm crit}$ ($\beta_{\rm crit}\sim0.69\pm0.02$ [14]) in the SOC model. In the SOC model, $\beta_{\rm crit}$ is measured in the ensemble average of the evolution for many segments. In other words, the growth of a given local segment does not proceed in a continuous way, and growth jumps among different segments. For any given segment, there are many waiting steps between two successive growths, and during the waiting times, growths occur at other segments. Considering the growth process in the DP model under the original definition of time, we find the DP model in the neighborhood of q_c is very similar to a local segment in the SOC model, that is, we can relate the DP system at $q\sim q_c$ to a local region embedded in a larger SOC system, while we relate the waiting times in the DP system to the growths of SOC system occurring in the regions away from the given local region. From all the above pictures, we can heuristically understand the reason that β_1 and β' in the DP model are about equal to the value of $\beta_{\rm crit}$ and $\beta_{\rm tr}$ of the SOC model, respectively.

To complete the discussion on how q influences the behavior of the system, we consider the extreme case, $q=0$. This is contrary to the opposite extreme case, $q=1$, where the behavior of the system is trivial because the interface cannot advance. As mentioned in Ref. [10], there is no scaling behavior at $q=0$. This conclusion has been confirmed by our simulations. But to our surprise, we find that the system has certain periodic properties. The result is presented in Fig. 5. From Fig. 5, we can find that the width of the interface $W(L,t)$ varies periodically while it increases (under the condition of Fig. 5, the whole increasing behavior is exponential), and the period is $30L$. According to our simulations the amplitude of the periodic oscillation is a constant except the initial stage. In Fig. 5, there are small fluctuations in various

periods, which is due to the fact that the site of growth is chosen randomly. If we average the curve for a large number of samples, we can obtain a smooth curve with very good periodicity (excluding the exponentially increasing tendency). To further investigate this phenomenon, we also simulate the dynamical system for small q. From the data, we find that (1) the period is independent of q and (2) the amplitude depends on q, and amplitude decreases when q increases. Furthermore, we find the period is also independent of the restricted difference of height between adjoint sites. In fact, this periodic behavior can be regarded as the result of the competition between events A and B. Generally speaking, events A roughen the interface while events B flatten it. Increasing A events may increase the probability for events B to appear, and vice versa. This crossing interaction mechanism makes A and B events dominant in turn, and leads to the appearance of the periodic behavior. Of course, this is an ambiguous statement. Some problems have not been well understood, i.e., what factors influence period? Why do events A and B control the width of the interface in such a periodic way? The mechanism of periodic behavior is still under study.

In summary, we have investigated the dynamical behavior of the DP model in detail. We find the width of the interface has the dynamical scaling behavior characterized by the form of Eq. (5). Furthermore, by introducing events A and B and a modified definition of time, we can interpret β_1 as an exponent describing the transient process and the exponent β_2 as an asymptotic exponent. The link between dynamical scaling behaviors of the DP model and the SOC model is briefly discussed. Finally, we find an interesting phenomenon that at $q=0$ and small q, the width of the interface $W(L,t)$ is a ''periodic'' oscillation of time added to an exponentially increasing behavior, and we heuristically explain the mechanism underlying this periodicity.

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